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ON INVESTIGATING RESONANT ALMOST-PERIODIC SYSTEMS FOR STABILITY WITH RESPECT TO A PART OF THE VARIABLES*

M.M. KHAPAEV and V.N. SHINKIN

A previously-uninvestigated case of fourth-order resonance of quasilinear systems with coefficients almost-peridic in time is examined. In this case the application of analytical methods of reduction to normal form faces a number of difficulties. This problem is solved below by means of a constructive construction of the perturbed Liapunov (Chetaev) function and of studying the extremal properties of the mean of its derivative /1/, containing only resonance terms.

Quasilinear systems have been studied in many papers devoted to the development of a number of qualitative ideas of the method of reduction to normal form in the sense of Briuno /2,3/, as well as in papers connected with the generalized second method of Liapunov /4,5/. Quasilinear systems with almost-periodic coefficients, formally reducible to autonomous ones, were studied in /6/ under resonance of odd order.

Consider the system

$$\begin{gathered} x_j^{\cdot} = \lambda_j y_j + f_{2j}'(t, x, y) + f_{3j}'(t, x, y) + F_{4j}'(t, x, y) \\ y_j^{\cdot} = -\lambda_j x_j + f_{2j}''(t, x, y) + f_{3j}''(t, x, y) + F_{4j}''(t, x, y) \\ x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_n), \quad j = 1, 2, \ldots, n \end{gathered} \tag{1}$$

We denote

$$\begin{gathered} z = (x_1, y_1, \ldots, x_n, y_n), \; f_2 = (f_{21}', f_{21}'', \ldots, f_{2n}', f_{2n}'') \\ f_3 = (f_{31}', f_{31}'', \ldots, f_{3n}', f_{3n}''), \quad F_4 = (F_{41}', F_{41}'', \ldots, F_{4n}', F_{4n}'') \end{gathered}$$

A) Let the right-hand sides of system (1) be analytic in z and be almost-periodic functions of time.

B) Let the functions $f_2(t, z)$ and $f_3(t, z)$ be polynomials of second and third degree in z, respectively, and let functions $F_4(t, z)$ be of higher than third order in z.

We consider as well the system

$$x_j^{\cdot} = \lambda_j y_j, \quad y_j^{\cdot} = -\lambda_j x_j, \quad j = 1, 2, \ldots, n \tag{2}$$

C) Let the eigenvalues $i\lambda_j$ $(i^2 = -1)$ of the linear part of system (2) be pure imaginary and not connected by resonance relations up to third order, inclusive.

Condition C signifies that any linear combination of the numbers $\pm\lambda_j$ with integer coefficients does not belong to the frequency spectrum of the coefficients of the original system (1) if the sum of the absolute values of all integers occurring as multipliers of $\pm\lambda_j$, does not exceed three.

System (2) has the Liapunov function

$$V_0(z, t) = \sum_{j=1}^{k} |\lambda_j| \frac{(x_j^2 + y_j^2)}{2}, \quad k \leqslant n$$

We construct the perturbed Liapunov function /1/ as a segment of a power series in z: $V = V_0 + S$, where the perturbation S is a homogeneous polynomial in z, in such a way that the total derivative of V relative to system (1) starts with terms of even order in z. We denote

$$\Psi(z, t) = \frac{\partial S}{\partial z} f_2 + \frac{\partial V_0}{\partial z} f_3, \quad H(z, t) = \frac{\partial V_0}{\partial z} f_2$$

We expand function H into a series in the complex variables $u_j = x_j + iy_j$, $v_j = x_j - iy_j$. By $\alpha_{m_1, \ldots, k_n}(t)$ we denote the coefficient in this series of the term $u_1^{m_1} v_1^{k_1} \ldots u_n^{m_n} v_n^{k_n}$ $(m_i, k_i$ are positive integers). We set

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$$\beta_{m_1\dots k_n}(t) = \exp\{-i(\lambda_1(m_1-k_1)+\dots+\lambda_n(m_n-k_n))t\} \times$$
$$\left[c_{m_1\dots k_n} - \int_0^t \alpha_{m_1\dots k_n}(t)\exp\{i(\lambda_1(m_1-k_1)+\dots+\lambda_n(m_n-k_n))t\}\,dt\right]$$
$$c_{m_1\dots k_n} = \mathrm{const}$$

We now construct the perturbation S as the series

$$S(x, y, t) = \sum_{m_1+\dots+k_n=3} \beta_{m_1\dots k_n}(t)(x_1+iy_1)^{m_1}\dots(x_n-iy_n)^{k_n}$$

We represent the general solution of system (2) in the form $x_j = r_j\cos(\lambda_j t+\theta_j)$, $y_j = r_j\sin(\lambda_j t+\theta_j)$. Hence respectively, $u_j = r_j\exp\{i(\lambda_j t+\theta_j)\}$, $v_j = r_j\exp\{-i(\lambda_j t+\theta_j)\}$. We compute the mean of Ψ along the solution of system (2)

$$\overline{\Psi}(r, \theta) = \lim_{t\to\infty}\frac{1}{t}\int_0^t \Psi(z(r, \theta, t), t)\,dt$$

By virtue of its construction Ψ is an almost-periodic function of time and, therefore, its mean $\overline{\Psi}$ always exists. It is convenient to compute the mean in the complex variables in which function Ψ takes the form

$$\Psi(u, v, t) = \sum_Q\sum_m \Psi_{Qm}r^Q\exp\{i((Q\Lambda)t+\Omega_m t+(Q\theta))\}$$
$$\Psi_{Qm}\in C, \quad \Lambda = (\lambda_1, -\lambda_1, \dots, \lambda_n, -\lambda_n),$$
$$\theta = (\theta_1, -\theta_1, \dots, \theta_n, -\theta_n)$$
$$\Omega_m\in R, \quad m\in Z, \quad Q = (q_1, \dots, q_{2n})\in Z^{2n},$$
$$r^Q = r_1^{q_1+q_2}\dots r_n^{q_{2n-1}+q_{2n}}$$

We shall say that internal resonance of order $|Q|$ is observed in system (1) if $(Q\Lambda)+\Omega_m = 0$. The internal resonance is said to be an identity resonance if $(Q\Lambda)\equiv 0$. Computing the mean $\overline{\Psi}$, we get that it contains only the resonance terms of function Ψ

$$\overline{\Psi}(r, \theta) = \sum_Q\sum_m \Psi_{Qm}r^Q\exp\{i(Q\theta)\}; \quad (Q\Lambda)+\Omega_m = 0$$

D) Let the mean $\overline{\Psi}(r, \theta)$ be a negative-definite function in $r_1, \dots, r_k$ $(k\leqslant n)$.

We formulate theorems on investigating for asymptotic stability and instability under a fourth-order identity resonance, based on the study of the extremal properties of the mean of the derivative of the perturbed Liapunov function.

Theorem 1. Let conditions A–D be fulfilled. Then the equilibrium position of system (1) is asymptotically stable with respect to a part $x_1, y_1, \dots, x_k, y_k$ of the variables.

Theorem 2. Let conditions A–C be fulfilled. Let the mean $\overline{\Psi}(r, \theta)$ be a positive-definite function in $r_1, \dots, r_k$ $(k\leqslant n)$. Then the equilibrium position of system (1) is unstable with respect to the part $x_1, y_1, \dots, x_k, y_k$ of the variables.

The proofs of Theorems 1 and 2 are analogous to the corresponding ones in /7/.

Let us now formulate a Chetaev-type theorem which is applicable to the study of instability under fourth-order internal resonance. The presence of nonidentity fourth-order resonances between the frequencies of linear system (2) can lead to sign-variability with respect to r of the quadratic form of mean $\overline{\Psi}(r, \theta)$. In this case, as V_0 we take a sign-variable integral of system (2), homogeneous in x, y. For this we consider the homogeneous function

$$V_0(r, \theta) = \sum_Q\sum_m G_{Qm}r^Q\exp\{i(Q\theta)\}, \quad G_{Qm}\in C, \quad (Q\Lambda)+\Omega_m = 0 \tag{3}$$

sign-variable in r. As $V_0(r, \theta)$ we can take, for instance, the mean $\overline{\Psi}(r, \theta)$. For r, θ we substitute their expressions in terms of x, y. The function $V_0(x, y, t) = V_0(r, \theta)$ obtained is precisely the required sign-variable integral of system (2). We construct the perturbed function $V = V_0 + S$ (just as we did above) in such a way that a total derivative of V relative to system (1) starts with even-order terms in x, y.

Theorem 3. Let conditions A–C be fulfilled. Let $V(z, t)$ be a positive-definite (negative-definite) function in the variables $x_1, y_1, \dots, x_k, y_k$ in the domain $V>0$ $(V<0)$. Let the mean $\overline{\Psi}(r, \theta)$ of the derivative of the perturbed function $V = V_0 + S$ (where V_0 is defined in (3)) be a positive-definite (negative-definite) function in $r_1, \dots, r_k$ in the domain $V(x, y, t) = V(r, \theta, t)>0$ (in the domain $V(x, y, t) = V(r, \theta, t)<0$). Then the equilibrium position of system (1) is unstable with respect to the part $x_1, y_1, \dots, x_k, y_k$ of the variables.

The proof of Theorem 3 relies on the results in /7/; therefore, we merely indicate the idea on which it is founded. Without loss of generality we take it that the mean $\overline{\Psi}$ is a positive-definite function in $r_1, \ldots, r_k$ in the domain $V > 0$. From /7/ it follows that $\exists \varepsilon_0 > 0$ and $\exists T_0 > 0$: for any solution $z(t)$, $z(t_0) = z_0$, such that $z_0 \in \{V > 0\}$, $|x_1| + |y_1| + \ldots + |x_k| + |y_k| \neq 0$, $|z_0| < \varepsilon_0$, it follows that $V(z(t), t) > V(z_0, t_0) > 0$ for $t > t_0 + T_0$. We take $\forall \varepsilon > 0$ $(\varepsilon \leqslant \varepsilon_0)$ and $\forall \delta \leqslant \varepsilon$. We select $L > 0$, such that the set $\{V(z, t) > L\}$ is located outside the $(\varepsilon + \Delta)$-neighborhood of zero with respect to the variables $x_1, y_1, \ldots, x_k, y_k$ $(\Delta > 0)$. From /7/ it follows that $\exists T_1 > T_0$: $V(z(t), t) > L$ for $t > t_0 + T_1$. Because function V is almost-periodic in t there exits $T_2 \geqslant T_1$ such that the set $\{V(z, t_0 + T_2) \geqslant L\}$ is located outside the $(\varepsilon + \Delta/2)$-neighborhood of zero with respect to the variables $x_1, y_1, \ldots, x_k, y_k$. Hence we get that $|z(t_0 + T_2)| \geqslant \varepsilon + \Delta/2 > \varepsilon$. Therefore, the zero solution of system (1) is unstable with respect to $x_1, y_1, \ldots, x_k, y_k$.

Note. When investigating third-order resonances in system (1), it is enough to set $S \equiv 0$ in Theorem 3 and to let the function $\Psi = H(z, t)$. When investigating resonances of higher than fourth order it is necessary, analogously to what was done above, to construct a Liapunov function in powers of z with due regard to terms of fourth, fifth, etc. orders, respectively.

Let us consider model examples of nonlinear systems with coefficients almost periodic in time and with a holomorphic right-hand side, in a neighborhood of the zero equilibrium position. Example 1 illustrates Theorem 1 on the asymptotic stability in the case of identity resonance

$$\begin{aligned}
x_1^{\cdot} = {} & \lambda_1 y_1 - x_1 y_1 3\sin t + 2y_1 y_2 (4\sin\sqrt{5}t + 1) - \lambda_1^{-1} x_1^2 \cos t - \\
& x_1^3 (2 + 3\cos\sqrt{2}t) + x_1 x_2^2 (\sin\sqrt{3}t + \cos\sqrt{5}t) + o(|z|^4) \\
y_1^{\cdot} = {} & -\lambda_1 x_1 - x_2^2 (2\sin\sqrt{2}t + 1) - \lambda_1^{-1} y_1 y_2 4\sqrt{5}\cos\sqrt{5}t + \\
& y_1 y_2^2 (3 + 7\cos t) + x_1 x_2 y_2 5\cos\sqrt{2}t + o(|z|^4) \\
x_2^{\cdot} = {} & -\lambda_2 y_2 - y_1^2 (4\sin\sqrt{5}t + 1) - \lambda_2^{-1} 2\sqrt{2} x_1 x_2 \cos\sqrt{2}t - x_1 x_2 y_1 (6 - \\
& \cos\sqrt{5}t) + o(|z|^4) \\
y_2^{\cdot} = {} & \lambda_2 x_2 + 2x_1 x_2 (2\sin\sqrt{2}t + 1) - y_1^2 y_2 (5 - 3\cos\sqrt{2}t) - y_2^3 (1 + \\
& \cos\sqrt{3}t) + o(|z|^4) \\
z = {} & (x_1, y_1, x_2, y_2)
\end{aligned} \tag{4}$$

We set $\lambda_1 = 2\sqrt{6}$, $\lambda_2 = 3\sqrt{6}$. Then the perturbed Liapunov function has a sufficiently simple form

$$\begin{aligned}
V = {} & 0.5\lambda_1 (x_1^2 + y_1^2) + 0.5\lambda_2 (x_2^2 + y_2^2) + x_1^3 \sin t + \\
& x_1 x_2^2 (2\sin\sqrt{2}t + 1) + y_1^2 y_2 (4\sin\sqrt{5}t + 1)
\end{aligned}$$

The mean $\overline{\Psi}$ of the derivative to the perturbed Liapunov function contains only the fourth-order identity resonance terms and is a sign-negative biquadratic function in r_i $(r_i^2 = x_i^2 + y_i^2,\ i = 1, 2)$ $\overline{\Psi} = \sqrt{6}\,(-1.5r_1^4 - 2.25r_1^2 r_2^2 - 1.125r_2^4)$.

Consequently, by Theorem 1 the zero equilibrium position of (4) is asymptotically stable.

Example 2 is on the investigation for instability in the case of an internal fourth-order resonance by use of the Chetaev-type Theorem 3

$$\begin{aligned}
x_1^{\cdot} = {} & \lambda_1 y_1 + (\sin t + \sqrt{5}\cos\sqrt{5}t)\, x_2^2 - \frac{5}{8}\lambda_2 (\cos t - \sin\sqrt{5}t)\, x_2 y_2 - \\
& \frac{3}{8}(\sin t + \sqrt{5}\cos\sqrt{5}t)\, y_2^2 + 2(2\lambda_1 + \lambda_2)\sin(2\lambda_1 + \lambda_2)\, t x_1^2 + \\
& 2\lambda_1 \cos(2\lambda_1 + \lambda_2)\, t x_1 y_1 + x_2^3 (1 - 5\sin\sqrt{7}t) + o(|z|^4) \\
y_1^{\cdot} = {} & -\lambda_1 x_1 - \frac{15}{8}\lambda_2 (\cos t - \sin\sqrt{5}t)\, x_2^2 - \frac{9}{8}(\sin t + \sqrt{5}\cos\sqrt{5}t)\, x_2 y_2 + \\
& 2\lambda_1 \sin(2\lambda_1 + \lambda_2)\, t x_1 y_1 + 2(2\lambda_1 + \lambda_2)\cos(2\lambda_1 + \lambda_2)\, t y_1^2 - \\
& y_1 x_2 y_2 (1 + \cos\sqrt{5}t) + x_1 y_2^2 \sin t + o(|z^4) \\
x_2^{\cdot} = {} & -\lambda_2 y_2 - (2\lambda_1 + \lambda_2)\cos(2\lambda_1 + \lambda_2)\, t y_1 x_2 - \lambda_2 \sin(2\lambda_1 + \lambda_2)\, t y_1 y_2 - \\
& x_1 y_1^2 (3\cos\sqrt{5}t - 1) + x_1 y_2^2 \sin\sqrt{7}t + o(|z|^4) \\
y_2^{\cdot} = {} & \lambda_2 x_2 - \lambda_2 \cos(2\lambda_1 + \lambda_2)\, t x_1 x_2 - (2\lambda_1 + \lambda_2)\sin(2\lambda_1 + \lambda_2)\, t x_1 y_2 - \\
& x_1^3 \sin\sqrt{5}t - x_2^3 (\cos t - 3\sin\sqrt{2}t) + o(|z|^4) \\
z = {} & (x_1, y_1, x_2, y_2)
\end{aligned} \tag{5}$$

A fourth-order internal resonance is observed in system (5) when $\lambda_1 = 3\sqrt{6}$, $\lambda_2 = \sqrt{6}$. In this case the perturbed Liapunov function is

$$\begin{aligned}
V = {} & x_1 x_2 (x_2^2 - 3y_2^2) - y_1 y_2 (3x_2^2 - y_2^2) + x_2^3 (\cos t - \sin\sqrt{5}t) - \\
& 2\sin(2\lambda_1 + \lambda_2)\, t y_1^2 y_2^3 + 2\cos(2\lambda_1 + \lambda_2)\, t x_1^2 x_2^3
\end{aligned}$$

The mean $\overline{\Psi}$ of the derivative of the perturbed Liapunov function is the sign-constant function $\overline{\Psi} = 0.125r_2^6$ positive definite in the domain $\{V > 0\}$. Consequently, by Theorem 3, the zero equilibrium position of system (5) is unstable.

We remark that with the use of a perturbed function we can investigate the instability of the Lagrange libration points of the circular restricted three-body problem under resonances of third and fourth orders, previously investigated by reduction to normal form in the sense of Briuno /8/. In the case of third-order resonance

$$V = V_0 + S,\ V_0 = x_1(x_2^2 - y_2^2) - 2y_1x_2y_2,\ S \equiv 0$$

In the case of fourth-order resonance

$$V_0 = x_1x_2(x_2^2 - 3y_2^2) - y_1y_2(3x_2^2 - y_2^2)$$

and S is uniquely determined by the formulas above.

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